SCHWARZ LEMMA FOR HARMONIC MAPPINGS IN THE UNIT BALL

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Abstract. We prove the following generalization of Schwarz lemma for harmonic mappings. If u is a harmonic mapping of the unit ball B_n onto itself such that $u(0) = 0$ and $\|u\|_p := \left(\int_S |u(\eta)|^p d\sigma(\eta)\right)^{1/p} < \infty$, $p \geqslant 1$ then $|u(x)| \leqslant g_p(|x|)\|u\|_p$ for some smooth sharp function g_p vanishing in 0. Moreover we provide sharp constant C_p in the inequality $\|Du(0)\| \leqslant C_p\|u\|_p$. Those two results extend some known result from harmonic mapping theory ([1, Chapter VI]).

1. Introduction

On the paper $\mathbf{R}^m$ is the standard Euclidean space with the norm $|x| = \sqrt{\sum x_i^2}$. Let $p \geqslant 1$ and assume that H^p is the Hardy space of the holomorphic mappings on the unit ball $B_n \subset \mathbf{C}^n \cong \mathbf{R}^{2n}$. In their classical paper [7], Macintyre and Rogosinski proved the following result: Let f be holomorphic on the unit disk such that $f(0) = 0$ and such that $\|f\|_{H^p} < \infty$ for $p \geqslant 1$, then

$$|f(z)| \leqslant \frac{|z|}{(1 - |z|^2)^{1/p}} \|f\|_{H^p} \tag{1.1}$$

with extremal functions $f(w) = \frac{Aw}{(1-\bar{z}w)^{2/p}}$. This is a generalization of Schwarz lemma (for $p = \infty$ it coincides with the classical Schwarz lemma). Then for holomorphic mappings on the unit ball $B_n \subset \mathbf{C}^n$ we have the following result of Zhu [8, Theorem 4.17]:

$$|f(z)| \leqslant \frac{1}{(1 - |z|^2)^{n/p}} \|f\|_p. \tag{1.2}$$

Let us sketch the proof of (1.2), which imply (1.1), for $n = 1$. By definition

$$\|f\|_p^p := \|f\|_{H^p}^p = \int_S |f(\eta)|^p d\sigma(\eta). \tag{1.3}$$

Here as $d\sigma$ is the normalized rotationally invariant Borel measure on the unit sphere $S = S_n = \partial B_n$. Choose the holomorphic change $\eta(w) = \varphi_z(w)$

2000 *Mathematics Subject Classification.* Primary 31A05; Secondary 42B30 .
Key words and phrases. Harmonic mappings, Schwarz inequality, Hardy space.

in (1.3), where φ_z is an automorphism of the unit ball such that $\varphi(0) = z$. Now by using holomorphic mapping

$$F_r(w) = f_r(\varphi_z(w)) \left(\frac{1-|z|^2}{(1-\langle z, w\rangle)^2}\right)^{n/p}, \tag{1.4}$$

and by making use of the mean value inequality we obtain (1.2). In order to derive (1.1) with $n = 1$, from (1.2), just alike for the classical proof of Schwarz lemma we make use of holomorphic mapping $g(z) = f(z)/z$, whose H^p norm coincides with the H^p norm of f. However an analogous inequality for higher-dimensional case ($n > 1$) cannot be proved in the same way. In this paper we will attack this problem for the class of harmonic mappings, which contain holomorphic mappings.

Assume now that that $1 \leqslant p \leqslant \infty$ and let $1/p + 1/q = 1$ and consider the Hardy class $\mathcal{H}^p$ of harmonic mappings defined in the unit ball, i.e. of harmonic mappings $f : B^n \to \mathbf{R}^m$ with

$$\|f\|_p := \sup_r \left(\int_S |f(r\eta)|^p d\sigma(\eta)\right)^{1/p} < \infty.$$

Here as before $d\sigma$ is the normalized rotationally invariant Borel measure on the unit sphere $S = S^{n-1}$.

It is well known that a harmonic function (and a mapping) $u \in \mathcal{H}^p(B)$, $p > 1$, where $B = B^n$ is the unit ball with the boundary $S = S^{n-1}$, has the following integral representation

$$u(x) = \mathcal{P}[f](x) = \int_{S^{n-1}} P(x, \zeta) f(\zeta) d\sigma(\zeta), \tag{1.5}$$

where

$$P(x, \zeta) = \frac{1-|x|^2}{|x-\zeta|^n}, \zeta \in S^{n-1}$$

is Poisson kernel and σ is the unique normalized rotation invariant Borel measure on S^{n-1} and $|\cdot|$ is the Euclidean norm.

Let us formulate the classical Schwarz lemma for harmonic mappings on the unit ball $B^n \subset \mathbf{R}^n$ and assume its image is $\mathbf{R}^m$. Let f be harmonic on the unit ball, and assume that $\|f\|_\infty < \infty$ and that $f(0) = 0$, then we have the following sharp inequality

$$|f(x)| \leqslant U(rN)\|f\|_\infty.$$

Here $r = |x|$, $N = (0, \dots, 0, 1)$ and U is a harmonic function of the unit ball into $[-1, 1]$ defined by

$$U(x) = \mathcal{P}[\chi_{S^+} - \chi_{S^-}](x), \tag{1.6}$$

where χ is the indicator function and $S^+ = \{x \in S : x_n \geqslant 0\}$, $S^- = \{x \in S : x_n \leqslant 0\}$.

Assume now that $p < \infty$. We are going to find a sharp function $g(r)$ satisfying the condition $g(0) = 0$ in the inequality

$$|f(x)| \leqslant g_p(r)\|f\|_p, \quad f \in \mathcal{H}^p, \quad f(0) = 0.$$

2. The main result

We prove the following theorem

Theorem 2.1. *Let $p \geqslant 1$, and let q be its conjugate and define*

$$g_p(r) = \inf_{a\in[0,\infty)} \left(\int_S |P_r(\eta) - a|^q d\sigma(\eta) \right)^{1/q}.$$

Then for $1 \leqslant p < \infty$, $g_p : [0,1) \to [0,\infty)$ is a smooth increasing diffeomorphism with $g_p(0) = 0$, and for every $f \in \mathcal{H}^p$ with $f(0) = 0$, we have

$$|f(x)| \leqslant g_p(|x|)\|f\|_p \tag{2.1}$$

and

$$\|Df(0)\| \leqslant n \left(\frac{\Gamma\left[\frac{n}{2}\right]\Gamma\left[\frac{1+q}{2}\right]}{\sqrt{\pi}\,\Gamma\left[\frac{n+q}{2}\right]} \right)^{\frac{1}{q}} \|f\|_p. \tag{2.2}$$

Both inequalities (2.1) and (2.2) are sharp. For $p = \infty$, we have $g_\infty(r) = U(rN)$ which coincides with Schwarz lemma and g_∞ is an increasing diffeomorphism of $[0,1]$ onto itself. Here $Df(0) : \mathbf{R}^n \to \mathbf{R}^m$ is the formal derivative and $\|Df(0)\| = \sup_{|h|=1} |Df(0)h|$.

Remark 2.2. It seems unlikely that we can explicitly express the function $g_p(r)$ for general p, however we demonstrate some special cases $p = 1, 2, \infty$ in Section 3, where among the other fact we prove the last part of this theorem. For some sharp pointwice estimates for the first derivative of harmonic mapping we refer to papers [2, 3, 6, 5]. Some optimal estimates of the harmonic function defined in the unit ball has been obtained in [4], but no normalization $f(0) = 0$ is imposed, so the obtained inequalities in [4] are not sharp in the context of this paper.

Proof. Let $x \in B_n$ and $\eta \in S^{n-1}$ with $r = |x|$ and let

$$P_x(\eta) = \frac{1 - r^2}{|x - \eta|^n}$$

and define

$$P_r(\eta) = \frac{1 - r^2}{(1 + r^2 - 2r\eta_n)^{n/2}}.$$

Then

$$f(x) = \int_S P_x(\eta) f(\eta) d\sigma(\eta),$$

where $S = S^{n-1}$ is the unit sphere. Now since $f(0) = 0$, it follows that

$$\int_S f(\eta) d\sigma(\eta) = 0,$$

and so

$$f(x) = \int_S (P_x(\eta) - a) f(\eta) d\sigma(\eta),$$

for a number $a = a(r)$ not depending on η.

Hence by using Hölder inequality, and unitary transformations of the unit sphere we have

$$|f(x)| \leqslant \left(\int_S |P_r(\eta) - a|^q d\sigma(\eta)\right)^{1/q} \|f\|_p. \tag{2.3}$$

So we are going to consider the minimum of the following function

$$\Phi_r(a) = \left(\int_S |P_r(\eta) - a|^q d\sigma(\eta)\right)^{1/q}.$$

Then $\Phi_r(a)$ is convex and it satisfies the conditions $\Phi_r'(0) < 0$ and $\Phi_r(\infty) = \infty$. This implies that there is a unique constant $a^* = a(r) \in (0, \infty)$ such that

$$\Phi_r(a^*) = \min_{a \in \mathbf{R}} \Phi_r(a).$$

To show that Φ_r is convex, observe the following simple fact $\Phi_r(a) = \|P_r - a\|_{L^q}$. So

$$\Phi_r(\lambda a+(1-\lambda)b) \leqslant \|\lambda(P_r-a)\|_{L^q}+\|(1-\lambda)(P_r-b)\|_{L^q} = \lambda\Phi_r(a)+(1-\lambda)\Phi_r(b).$$

In order to prove that $\Phi_r'(0) < 0$, by calculation we find out that

$$\Phi_r'(0) = -\int_S |P_r|^{q-1} d\sigma(\eta) \left(\int_S |P_r|^q d\sigma(\eta)\right)^{1/q-1} < 0.$$

Furthermore

$$\Phi_r'(a) = -\left(\int_S |P_r(\eta) - a|^q d\sigma(\eta)\right)^{1/q-1} F(r, a), \tag{2.4}$$

where

$$F(r, a) = \int_S (P_r(\eta) - a)|P_r(\eta) - a|^{q-2} d\sigma(\eta).$$

So

$$F_a(r, a) = (q-1)\int_S |P_r(\eta) - a|^{q-2} d\sigma(\eta) > 0,$$

and this implies in particular that $F(r, a)$ as a function of a is strictly increasing, so Φ_r has only one stationary point which is its minimum which we denote by $a^* = a(r)$.

Since

$$F(r, a(r)) = 0, \tag{2.5}$$

the implicit function theorem implies that there is a smooth function a^* depending on r such that

$$\frac{\partial a(r)}{\partial r} = -\frac{\frac{\partial F}{\partial r}}{\frac{\partial F}{\partial a}}.$$

Thus the function $g_p(r) = \Phi_r(a(r))$ is smooth function of r as a composition of smooth functions. It satisfies the condition $g_p(0) = 0$ and we have

$$|f(x)| \leqslant g_p(|x|)\|f\|_{L^p}.$$

Moreover

$$g_p'(0) = \left(\int_S |n\eta_n|^q d\sigma(\eta)\right)^{1/q} = n\left(\frac{\Gamma\left[\frac{n}{2}\right]\Gamma\left[\frac{1+q}{2}\right]}{\sqrt{\pi}\Gamma\left[\frac{n+q}{2}\right]}\right)^{\frac{1}{q}}$$

So

$$\|Df(0)\| \leqslant n\left(\frac{\Gamma\left[\frac{n}{2}\right]\Gamma\left[\frac{1+q}{2}\right]}{\sqrt{\pi}\Gamma\left[\frac{n+q}{2}\right]}\right)^{\frac{1}{q}}$$

Since $g_p(r) = \Phi(r, a(r))$ we have

$$\partial_r g_p(r) = \partial_1\Phi(r, a(r)) + \partial_2\Phi(r, a(r))\partial_r a(r) = \partial_1\Phi(r, a(r)).$$

Since

$$\Phi(r, a) = \left(\int_S |P_r(\eta) - a|^q \, d\sigma(\eta)\right)^{1/q},$$

we have that

$$\partial_1\Phi(r, a) = \int_S \partial_r P_r(\eta)(P_r(\eta) - a)|P_r(\eta) - a|^{q-2} d\sigma(\eta)\left(\int_S |P_r(\eta) - a|^q \, d\sigma(\eta)\right)^{1/q-1}.$$

Since

$$P_r(\eta) = \frac{1 - r^2}{(1 + r^2 - 2r\eta_n)^{n/2}},$$

it follows that

$$P_r(\eta) - a = \frac{1 - r^2}{(1 + r^2 - 2r\eta_n)^{n/2}} - a = n\eta_n r + (1 - a) + O(r^2),$$

and hence

$$\partial_1\Phi(r, 1) = O(r) + \left(\int_S |n\eta_n|^q d\sigma(\eta)\right)^{1/q}.$$

Next we have $\lim_{r\to 0}\partial_1\Phi(r, a(r)) = \partial_a\Phi(0, 1)$, and so

$$\partial_r g_p(0) = \left(\int_S |n\eta_n|^q d\sigma(\eta)\right)^{1/q}.$$

In order to show that the inequality is sharp, fix x and without loosing the generality assume that $x = RN$, with $R = |x|$ and let

$$f_R(\eta) = |P_R(\eta) - a(R)|^{q/p}\mathrm{sign}(P_R(\eta) - a(R))$$

and let $u_R(y) = P[f_R](y)$. From (2.5)

$$F(R, a) = \int_S (P_R(\eta) - a(R))|P_R(\eta) - a(R)|^{q-2} dt = 0.$$

Hence we obtain that $u_R(0) = 0$. Now, the Hölder inequality (2.3) is an equality for u_R in x. This implies the sharpness of inequality. In order to prove that for $p < \infty$, $\lim_{r\to 1} g_p(r) = \infty$, let $f \in \mathcal{H}_p \setminus \mathcal{H}_\infty$ and assume without loosing of generality that $f(0) = 0$. Then

$$\sup_r g_p(r) \geqslant \frac{\sup_x |f(x)|}{\|f\|_p} = \infty.$$

Finally prove that g_p is strictly increasing. Let $r < s$ and choose $\|f\|_p = 1$ such that $g_p(r) = |f(x_0)| = \max_{|x|\leqslant r} |f(x)|$. Clearly f is not a constant function. Then by maximum principle $|f(x_0)| < \max_{|x|=s} |f(x)| \leqslant g_p(s)$. So g_p is a strictly increasing function.

The last part of the proof, i.e. the case $p = \infty$, follows from the previous proof and the next section. □

As a corollary of our main result we obtain

Corollary 2.3. *If $f \in \mathcal{H}^2$, then $\|Df(0)\| \leqslant \sqrt{n}\sqrt{\|f\|_2^2 - |f(0)|^2}$.*

Proof. Let $g(x) = f(x) - f(0)$, then $Df(0) = Dg(0)$, on the other hand

$$\|g\|_2^2 = \langle f - f(0), f - f(0)\rangle = \|f\|^2 + |f(0)|^2 - 2\langle f, f(0)\rangle = \|f\|_2^2 - |f(0)|^2.$$

On the other hand

$$|Dg(0)| \leqslant \lim_{r\to 0} g_2'(r)\|g\|_2 = \sqrt{n}\|g\|_2.$$

The result follows. □

3. Special cases

3.1. **The case $p = \infty$.** In this case we deal with the extremal problem

$$\inf_a \int_S |P_r(\eta) - a| d\sigma(\eta).$$

Let $a_0 = \frac{1-r^2}{(1+r^2)^{n/2}}$. Then

$$\begin{aligned}
\int_S |P_r(\eta) - a_0| d\sigma(\eta) &= \int_{S^+} P_r(\eta) d\sigma(\eta) - \int_{S^-} P_r(\eta) d\sigma(\eta) \\
&= \int_{S^+} (P_r(\eta) - a) d\sigma(\eta) - \int_{S^-} (P_r(\eta) - a) d\sigma(\eta) \\
&\leqslant \inf_a \int_S |P_r(\eta) - a_0| d\sigma(\eta).
\end{aligned}$$

So $a^* = \frac{1-r^2}{(1+r^2)^{n/2}}$.

This implies that

$$\|f(z)\| \leqslant U(rN)\|f\|_\infty,$$

which is known as the classical Schwarz lemma for harmonic mappings.

3.2. **The case $p = 2$.** In this case we deal with the extremal problem

$$g_p(r) = \left(\inf_a \int_S |P_r(\eta) - a|^2 d\sigma(\eta)\right)^{1/2}.$$

We have (see [1, p. 140])

$$\int_S |P_r(\eta) - a|^2 d\sigma(\eta) = \int_S P_r^2(\eta) d\sigma(\eta) + a^2 \int_S d\sigma(\eta) - 2a \int_S P_r(\eta) d\sigma(\eta)$$

$$= \frac{1 - |x|^4}{(1 - 2|x|^2 + |x|^4)^{n/2}} + a^2 - 2a.$$

So $a^* = 1$ and

$$\left(\inf_a \int_S |P_r(\eta) - a|^2 d\sigma(\eta)\right)^{1/2} = \sqrt{\frac{1 + |x|^2}{(1 - |x|^2)^{n-1}} - 1}.$$

This implies that

$$|f(x)| \leqslant \left(\sqrt{\frac{1 + r^2}{(1 - r^2)^{n-1}} - 1}\right) \|f\|_2$$

which coincides with analogous statement in [1, p. 140].

3.3. **The case $p = 1$.** In this case we deal with the extremal problem

$$g_p(r) = \inf_a \sup_\eta |P_r(\eta) - a|.$$

Since

$$\max_\eta P_r(\eta) = \frac{1 - r^2}{(1 - r)^n}$$

and

$$\min_\eta P_r(\eta) = \frac{1 - r^2}{(1 + r)^n},$$

we easily conclude that

$$g_p(r) = \frac{1}{2}\left(\frac{1 - r^2}{(1 - r)^n} - \frac{1 - r^2}{(1 + r)^n}\right)$$

(In this case $a^* = \frac{1}{2}\left(\frac{1-r^2}{(1-r)^n} + \frac{1-r^2}{(1+r)^n}\right)$.) So

$$|f(x)| \leqslant \frac{1}{2}\left(\frac{1 - r^2}{(1 - r)^n} - \frac{1 - r^2}{(1 + r)^n}\right) \|f\|_1.$$

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